In this section, *The American Statistician* publishes articles and notes of interest to teachers of the first mathematical statistics course and of applied statistics courses. To be suitable for this section, articles and notes should be useful to a substantial number of teachers of such a course or should have the potential for fundamentally affecting the way in which the course is taught.

Multiple Comparisons in a Mixed Model

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In this note we give a simple proof of the result that, in the symmetric version of Scheffé's mixed model for a balanced two-way layout, an exact T procedure for pairwise comparisons between the levels of the fixed factor can be based on the interaction mean square as the Studentizing factor. Such a proof does not seem to be available in the literature and will hopefully remove the confusion caused by some textbooks that incorrectly prescribe the error mean square as the Studentizing factor.

KEY WORDS: Multiple comparisons; Tukey procedure; Mixed two-way layout; Intraclass correlation; Studentized range distribution.

1. INTRODUCTION

We consider Sheffé's (1956,1959) mixed model for the analysis of variance of a two-way layout with one factor (say, A) fixed, the other factor (say, B) random, and with equal number of observations per cell. Scheffé (1959, p. 270) mentions that generally the ratio MS_A/MS_{AB} can be used only as an approximate F statistic for testing the null hypothesis for factor A. Scheffé gives an exact test and the associated simultaneous confidence intervals for all contrasts among the levels of factor A using Hotelling's T^2 test. The resulting confidence intervals are too conservative if the experimenter is interested only in pairwise comparisons. This leads us to consider a Tukey (T) procedure based on the Studentized range distribution. Scheffé (1959, p. 271) describes such a T procedure, which uses MS_{AB} as the Studentizing factor in the denominator; in a footnote on page 270 he also mentions that under a "symmetric" version of his model (see Section 2 for more details) the usual F test just mentioned and the T procedure are exact. No proof is given, however.

We have two objectives in presenting this note. The first is to give a simple proof of this exact result. The proof is probably known to some people but, to our knowledge, has not appeared in print. Our second objective is to remove the confusion caused by some textbooks (namely, Gibra 1973, p. 397) that incorrectly prescribe MS_E, the error mean square, as the Studentizing factor to be used in the T procedure. The correct Studentizing factor is the interaction mean square, MS_{AB}, regardless of which one of the three common models (Hocking 1973) for mixed two-way layout is employed. The idea of the proof is of some intrinsic value because it can be easily extended to higher-way balanced mixed models and as such can be introduced in a linear models course.

2. MODEL AND PRELIMINARIES

We shall not go into the details of how Scheffé arrives at his mixed model. Instead, we give the model in a final form that is most convenient for our purposes; this model is also given as Model Ia by Hocking (1973). Let Y_{ijk} be the kth observation on the ith level of factor A and the jth level of factor B ($1 \le i \le I$, $1 \le j \le J$, $1 \le k \le K$). We assume that the Y_{ijk} are jointly normally distributed with a covariance structure that depends on the error variance σ_E^2 and an $I \times I$ positive definite, symmetric matrix $\boldsymbol{\Sigma} = ((\sigma_{ii'}))$ both of which are assumed unknown. The specific model is

$$\begin{aligned} E(Y_{ijk}) &= \mu_i, \\ \operatorname{cov}(Y_{ijk}, Y_{i'j'k'}) &= \sigma_{ii} + \sigma_E^2 && \text{if } i = i',\ j = j',\ k = k' \\ &= \sigma_{ii} && \text{if } i = i',\ j = j',\ k \ne k' \\ &= \sigma_{ii'} && \text{if } i \ne i',\ j = j' \\ &= 0 && \text{if } j \ne j'. \end{aligned} \tag{2.1}$$

Scheffé's symmetric model assumes that the σ_{ii} are all equal to, say, σ^2, and the $\sigma_{ii'}$ for $i \ne i'$ are all equal to, say, $\rho\sigma^2$ with $-1/(I-1) \le \rho \le 1$. (Models II and III in Hocking (1973) are special cases of the symmetric Scheffé model for $\rho > 0$.) Henceforth we restrict our-

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selves to this symmetric version of (2.1), which can be stated as follows:

$$
\begin{aligned}
E(Y_{ijk}) &= \mu_i, \\
\operatorname{var}(Y_{ijk}) &= \sigma^2 + \sigma_E^2 = \sigma_Y^2 \quad \text{(say)}, \\
\operatorname{corr}(Y_{ijk}, Y_{i'j'k'}) &= \rho_1 = \frac{\sigma^2}{\sigma_Y^2} \quad \text{if } i = i', j = j', k \neq k' \\
&= \rho_2 = \frac{\rho\sigma^2}{\sigma_Y^2} \quad \text{if } i \neq i', j = j' \\
&= 0 \qquad\quad \text{if } j \neq j',
\end{aligned}
\tag{2.2}
$$

where all the parameters are unknown.

Our objective is to derive the T procedure for making all pairwise comparisons $\mu_i - \mu_{i'}$ $(1 \leq i < i' \leq I)$. Although we give the proof only for the T procedure, the same results can also be used to show the exactness of the F test. Huynh and Feldt (1970) have used a different technique to show the F distribution of certain mean square ratios in repeated measurements designs.

3. DERIVATION OF THE T PROCEDURE

The case of no replication, that is, $K = 1$, which leads to an intraclass correlation model among the Y_{ij} for fixed j, has been dealt with by Bhargava and Srivastava (1973). When we have multiple observations per cell we get an intraclass correlation model with two different correlation coefficients among the Y_{ijk}, as can be seen from (2.2). Therefore, the method of proof used by Bhargava and Srivastava must be applied iteratively in two steps.

In the following we use the usual dot notation to denote the average taken over the dotted subscripts, that is, $\overline{Y}_{i..} = \sum_{j,k} Y_{ijk}/JK$, $\overline{Y}_{ij.} = \sum_k Y_{ijk}/K$, and so on. Also we use the usual formulas for the SS's and the MS's in the ANOVA table; in particular, $MS_{AB} = SS_{AB}/(I-1)(J-1)$, where $SS_{AB} = K \sum_{i,j} (\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...})^2$. Finally, let $Q_{p,\nu}$ denote a Studentized range variable, which is the ratio of the range of p iid $N(0,1)$ variables and an independently distributed $(\chi_\nu^2/\nu)^{1/2}$ variable, and let $Q_{p,\nu}^{(\alpha)}$ denote the upper α point of its distribution.

Theorem. Under model (2.2) the random variable

$$
\max_{1 \leq i < i' \leq I} \frac{|\overline{Y}_{i..} - \overline{Y}_{i'..} - (\mu_i - \mu_{i'})|}{\sqrt{MS_{AB}/JK}} \tag{3.1}
$$

is distributed as a Studentized range variable $Q_{I,(I-1)(J-1)}$, and hence the simultaneous $100(1-\alpha)\%$ confidence intervals for $\mu_i - \mu_{i'}$ $(1 \leq i < i' \leq I)$ are given by the probability statement

$$
P\left\{\mu_i - \mu_{i'} \in \left[\overline{Y}_{i..} - \overline{Y}_{i'..} \pm Q_{I,(I-1)(J-1)}^{(\alpha)} \sqrt{\frac{MS_{AB}}{JK}}\right],\ \forall i < i'\right\} = 1 - \alpha. \tag{3.2}
$$

Proof. First make the transformation

$$
X_{ijk} = Y_{ijk} - b\overline{Y}_{.j.}, \tag{3.3}
$$

where the real constant b is chosen so that the X_{ijk} for the same j and different i are independent. (Of course, X_{ijk} and $X_{i'j'k'}$ for $j \neq j'$ are independent for any choice of b.) That is, b solves the quadratic equation

$$
\begin{aligned}
&\operatorname{cov}(X_{ijk}, X_{i'jk'}) \quad \text{for } i \neq i' \\
&= \left\{\rho_2 - \frac{2b}{IK}\left[1 + \rho_1(K-1) + \rho_2 K(I-1)\right] + \frac{b^2}{IK}\left[1 + \rho_1(K-1) + \rho_2 K(I-1)\right]\right\}\sigma_Y^2 \\
&= 0,
\end{aligned}
$$

and the solution can be verified to be real by using the fact that $\rho_1 \geq \rho_2$. Next make the transformation,

$$
\begin{aligned}
Z_{ijk} &= X_{ijk} - c\overline{X}_{ij.} \\
&= Y_{ijk} - b(1-c)\overline{Y}_{.j.} - c\overline{Y}_{ij.},
\end{aligned}
\tag{3.4}
$$

where now the real constant c is chosen so that the Z_{ijk} for the same j and same i are independent. (Of course, Z_{ijk} and $Z_{i'j'k'}$ for $i \neq i'$ or $j \neq j'$ are independent for any choice of c.) That is, c solves the quadratic equation

$$
\begin{aligned}
&\operatorname{cov}(Z_{ijk}, Z_{ijk'}) \quad \text{for } k \neq k' \\
&= \left\{(\rho_1 - \rho_2) - \frac{2c}{K}\left[\frac{\sigma_X^2}{\sigma_Y^2} + (K-1)(\rho_1 - \rho_2)\right] + \frac{c^2}{K}\left[\frac{\sigma_X^2}{\sigma_Y^2} + (K-1)(\rho_1 - \rho_2)\right]\right\}\sigma_Y^2 \\
&= 0,
\end{aligned}
$$

where $\sigma_X^2 = \operatorname{var}(X_{ijk})$, and the solution to this quadratic equation can also be verified to be real.

The transformations (3.3) and (3.4) make the Z_{ijk} mutually independent normal variables with $E(Z_{ijk}) = (1-c)(\mu_i - b\overline{\mu}_.)$ (from (3.4)) and $\operatorname{var}(Z_{ijk}) = \sigma_Z^2$ (say), which is the same for all i, j, k. The Z_{ijk} then follow the usual fixed-effects balanced one-way layout model. This fact along with (3.4) gives the following:

$$
\begin{aligned}
&\overline{Z}_{i..} - (1-c)(\mu_i - b\overline{\mu}_.) \\
&\quad = (1-c)(\overline{Y}_{i..} - \mu_i - b(\overline{Y}_{...} - \overline{\mu}_.)) \\
&\quad \sim N\left(0, \frac{\sigma_Z^2}{JK}\right) \ (1 \leq i \leq I),
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
SS_B^Z &= IK \sum_j (\overline{Z}_{.j.} - \overline{Z}_{...})^2 \\
&= (1-b)^2(1-c)^2\, SS_B \sim \sigma_Z^2 \chi_{J-1}^2,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
SS_{AB}^Z &= K \sum_{i,j} (\overline{Z}_{ij.} - \overline{Z}_{i..} - \overline{Z}_{.j.} + \overline{Z}_{...})^2 \\
&= (1-c)^2\, SS_{AB} \sim \sigma_Z^2 \chi_{(I-1)(J-1)}^2,
\end{aligned}
\tag{3.7}
$$

$$
SS_E^Z = \sum_{i,j,k} (Z_{ijk} - \overline{Z}_{ij.})^2 = SS_E \sim \sigma_Z^2 \chi_{IJ(K-1)}^2, \tag{3.8}
$$

where the SS's without superscripts are the SS's in terms of the Y's. Note that (3.5)–(3.8) are independently distributed of each other. Since the Z's are unobservable while the Y's are observable, and since b and c are constants that depend on unknown parameters, it is clear after an examination of (3.5)–(3.8) that, although σ_Z^2 can also be estimated from SS_B^Z and SS_E^Z, only SS_{AB}^Z can be used to form the Studentized range statistic

$$\max_{1 \le i < i' \le I} \frac{|\bar{Z}_{i..} - \bar{Z}_{i'..} - (1-c)(\mu_i - \mu_{i'})|}{\left\{SS_{AB}^Z/JK(I-1)(J-1)\right\}^{1/2}}$$

$$= \max_{1 \le i < i' \le I} \frac{(1-c)|\bar{Y}_{i..} - \bar{Y}_{i'..} - (\mu_i - \mu_{i'})|}{(1-c)\sqrt{MS_{AB}/JK}}$$

$$\sim Q_{I,(I-1)(J-1)},$$

which proves (3.1). The result (3.2) follows immediately.

Corollary. Simultaneous confidence intervals for all contrasts $\sum_i c_i\mu_i$, where $\sum_i c_i = 0$, are given by the probability statement

$$P\left\{\sum_i c_i\mu_i \in \left[\sum_i c_i\bar{Y}_{i..} \pm Q_{I,(I-1)(J-1)}^{(\alpha)} \sqrt{\frac{MS_{AB}}{JK}} \times \sum_i \frac{|c_i|}{2}\right] \text{ for all contrasts}\right\} = 1 - \alpha.$$

Proof. Use Lemma 1 of Miller (1966, p. 44).

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The Geometry of Rank-Order Tests

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This article examines the geometry of rank-order tests. We show that the set of rankings of n objects can be represented as the extreme points of a polyhedron determined by a set of linear constraints. Various rank-order statistics are interpreted via this geometric model. The model allows a unified presentation and illustrates the mechanics of rank-order tests.

1. INTRODUCTION

Although the development of distribution-free statistical tests can be traced as far back as 1710, the basis for many of the best-known distribution-free tests is Fisher's (1951) method of randomization. If applied directly to the original observations it produces efficient but impractical tests. However, the sample space for the test statistics can be standardized with the replacement of the original observations by their ranks. The resulting rank-order tests maintain much of the high efficiency while becoming vastly superior in practicality and ease of application. The statistical efficiency, ease, speed, and scope of application, however, only partly account for the success of distribution-free rank-order tests. If the data available relate solely to order or deal with a qualitative characteristic that can be ranked but not measured, the use of rank-order tests is inescapable.

In this article, we investigate rankings and statistics based on rankings. We show that the space of rankings can be characterized algebraically by a set of linear constraints. The resulting polyhedron is a geometric model in which the interpretation of various rank-order statistics becomes exceedingly transparent. To the authors' knowledge, despite the fact that rank-order tests are widely employed, this lucid interpretation of rank-order statistics has (with one exception (Schulman 1979)) apparently been ignored. The geometric model presented in Section 2 has many advantages not present in other characterizations. It captures the discrete set of rankings as the extreme points of a polyhedron determined by a set of linear constraints. The spacial visualization afforded by this model should aid in the development of improved rank-order models. In addition it has proven, in the authors' experience, to be a most effective pedagogical tool, in that it allows for a unified representation of many familiar distribution-free tests.

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